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COMPLETE UNIVERSAL LOCALLY FINITE GROUPS

BY

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ABSTRACT. This paper will partly strengthen a recent application of model theory to the construction of sets of pairwise nonembeddable universal locally finite groups [8]. Our result is

THEOREM. *There is a set* $\mathfrak{U}$ *of* $2^{\aleph_1}$ *universal locally finite groups of order* $\aleph_1$ *with the following properties*:

0.1. *If* $U \neq V \in \mathfrak{U}$ *and* A *and* B *are uncountable sugroups of* U *and* V, *then* A *and* B *are not isomorphic.*

Let A *be an uncountable subgroup of* $U \in \mathfrak{U}$.

0.2. A *does not belong to any proper variety of groups, and*

0.3. A *is not isomorphic to any of its proper subgroups.*

0.4. *Every* $U \in \mathfrak{U}$ *is a complete group* (*every automorphism of* U *is inner*).

NOTATIONS. $S < T$ means S is properly contained in T;

${}^{o}S = |S|$ is the cardinal (order) of the set S;

Aut G and Inn G are respectively the groups of automorphisms and inner automorphisms of the group G;

ISO(G, H) is the set of isomorphisms of G onto H;

If $S \leqslant G$, then gp(S) is the smallest subgroup of G containing S;

ω_1 is the first uncountable ordinal;

An *amalgam* A is the union of two groups, $A = F \cup H$, where F and H meet in a common subgroup, $F \cap H = E$, and we write

$$A = \begin{array}{c} F \quad H \\ \searrow\!\swarrow \\ E \end{array}$$

Introduction. A group U is in the class ULF of universal locally finite groups if

$ULF0$. U is locally finite (l.f.),

$ULF1$. Every finite group is isomorphic to a subgroup of U, and

$ULF2$. If E and F are isomorphic finite subgroups of U, then there is some $\varphi \in$ Aut U such that $\varphi(E) = F$.

From these follows

ULF 3. If $A \leqslant B$ are finite groups and η: $A \to U$ is an embedding, then

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there is an embedding $g\colon B \to U$ such that $g(A) = \eta(A)$. For, by *ULF*1 there is an embedding $f\colon B \to U$; and by *ULF*2 there is $\alpha \in \operatorname{Aut} U$ such that $\alpha f(A) = \eta(A)$. Thus $g = \alpha f$.

Philip Hall introduced the concept of a *ULF* group in [3] and showed, among other things,

0.5. There is a countable *ULF* group C which is unique up to isomorphism and

0.6. Every l.f. group of order κ is contained in a *ULF* group of order $\max(\kappa, \aleph_0)$. In particular, every countable l.f. group is embedded in C.

Hall constructed C as a tower of finite symmetric groups with regular embeddings. The regular representation $L < S(L)$, where $S(L)$ is the symmetric group on L, has the property

0.7. If E and F are subgroups of L and $\varphi \in \mathrm{ISO}(E, F)$, then for some $x \in S(L)$, $\varphi(a) = x^{-1}ax$ for all $a \in E$.

It follows that *ULF*2 can be replaced by the more useful

*ULF*4. If E and F are finite subgroups of U and $\varphi \in \mathrm{ISO}(E, F)$, then for some $x \in U$, $\varphi(a) = x^{-1}ax$ for all $a \in E$. For, let $G = \mathrm{gp}(E, F)$. By *ULF*3 let $g\colon S(G) \to U$ be an embedding such that $g(G) = G$ where $G < S(G)$ is the regular embedding. By 0.7 there is an $x \in g(S(G))$ as required.

Similarly, *ULF*3 can be strengthened to

*ULF*5. If $A \leqslant B$ are finite groups and $\eta\colon A \to U$ is an embedding then η can be extended to an embedding $g\colon B \to U$.

Hall's group C is also obtained in the more general algebraic construction of Jónsson [5] and model theoretic construction of Morley and Vaught [10].

More recently, answering a question of Kegel and Wehrfritz [6], Macintyre and Shelah proved [8]

0.8. For every $\kappa > \aleph_0$ there are 2^κ isomorphism types of *ULF* groups of order κ and

0.9. If $\kappa > \aleph_0$ is regular, there are 2^κ pairwise nonembeddable *ULF* groups of order κ.

Their proof uses results on unsuperstable first order theories in infinitary logic. In another paper [9] Macintyre gives more information on a question posed in [6]. A l.f. group H is *inevitable* if H is a subgroup of every *ULF* group of order $\geqslant {}^oH$. Hall proved every countable group to be inevitable. Macintyre proves that there is a *ULF* group of order $\aleph_1$ all of whose abelian subgroups are countable. Hence there are no inevitable abelian groups of order $\aleph_1$.

The present result strengthens 0.9 for the case $\kappa = \aleph_1$. A consequence of 0.1 is that no group of order $\aleph_1$ is inevitable. The existence of complete *ULF* groups contrasts sharply with the fact that Aut C contains a free group on $2^{\aleph_0}$ generators (which will not be proved).

We will note some basic properties of the class *ULF*.

0.10. Every countable subset of a *ULF* group is contained in a subgroup isomorphic to Hall's group C.

0.11. If every finite subset of a group G is contained in a *ULF* subgroup of G, then $G \in ULF$.

The proofs of both use *ULF*4. For 0.10 the uniqueness of C is also used.

1. **The basic construction.** This section gives a general procedure for constructing *ULF* groups of order $\aleph_1$, and subsequent sections add specifics to the construction so that the groups obtained satisfy 0.1–0.4.

The construction requires an amalgamation procedure for finite groups. The following is easily proved using the permutational product of B. H. Neumann [11].

1.1. LEMMA. *Suppose E, F, and H are finite groups, $E < H$, and g: $E \to F$ is a proper embedding. Then there exists a finite group $G > H$ and an embedding φ: $F \to G$ such that φg is the identity on E, $\varphi F \cap H = E$, and $G = \mathrm{gp}(\varphi F, H)$. Put differently, there is a finite group $G = \mathrm{gp}(A)$ where*

$$A = \begin{matrix} \varphi F & & H \\ & \diagdown\!\diagup & \\ & E & \end{matrix}$$

where φ is an isomorphism such that $\varphi^{-1} \equiv g$ on E.

Outline of the construction. For each ordinal $\alpha < \omega_1$ let $C_\alpha \cong C$ (Hall's group). We will define proper embeddings f_α: $C_\alpha \to C_{\alpha+1}$ obtaining a tower $\cdots \tilde{C}_\alpha < \tilde{C}_{\alpha+1} \cdots$ (via f_α, where $\sim$ is the direct limit map, with continuity at limit α). By 0.11 each limit group $\tilde{C}_\alpha \cong C$, and the union U of the entire tower is a *ULF* group of order $\aleph_1$.

The construction of f_α. Let $1 < F_1 < \cdots < F_n \cdots$ be finite subgroups of C_α such that $C_\alpha = \bigcup F_n$. We will define a chain of embeddings φ_n: $F_n \to C_{\alpha+1}$ and put $f_\alpha = \bigcup \varphi_n$. Let $F_0 = 1$ and φ_0: $F_0 \to C_{\alpha+1}$. Assume φ_i has been defined for all $i \leqslant n$. Choose a finite subgroup H_n of $C_{\alpha+1}$ such that $E_n = \varphi_n(F_n) < H_n$ and $H_i < H_n$ for all $i < n$ (H_i has been chosen previously). The chain $\{H_n\}$ will be chosen so that $C_{\alpha+1} = \bigcup H_n$. This can be done by arranging that H_n contain the nth member of some list of $C_{\alpha+1}$.

We now "amalgamate F_{n+1} and H_n via φ_n", that is, the subgroups F_n and $\varphi_n(F_n) = E_n$ are identified. Specifically, using Lemma 1.1, we obtain

1.2. $G = G_n = \mathrm{gp}(A)$ where

$$A = \begin{matrix} \varphi F_{n+1} & & H_n \\ & \diagdown\!\diagup & \\ & E_n & \end{matrix}$$

and

1.3. $\varphi \equiv \varphi_n$ on F_n.

By *ULF*5, $\eta =$ the identity on H_n can be extended to an embedding τ: $G \to C_{\alpha+1}$. We define $\varphi_{n+1} = \tau\varphi$: $F_{n+1} \to C_{\alpha+1}$ and observe

1.4. $\varphi_{n+1} \equiv \varphi_n$ on F_n, which follows from 1.3 and the fact that $\tau \equiv 1$ on $H_n > E_n$.

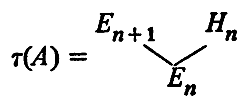
Thus there is a unique embedding f_α: $C_\alpha \to C_{\alpha+1}$ such that $f_\alpha \equiv \varphi_n$ on F_n for all $n \geqslant 0$.

We will list the main properties of f_α for later use.

1.5. *Summary of notation.* $C_\alpha = \bigcup F_n$, $C_{\alpha+1} = \bigcup H_n$, f_α: $C_\alpha \to C_{\alpha+1}$, $f_\alpha \equiv \varphi_n$ on F_n, $E_n = \varphi_n(F_n) = f_\alpha(F_n) < H_n$ and so $f_\alpha(C_\alpha) = \bigcup E_n < C_{\alpha+1}$.

1.6. $H_n \cap E_{n+1} = E_n$ for all $n \geqslant 0$.

PROOF. Since $\tau \equiv 1$ on H_n, from 1.2 we have

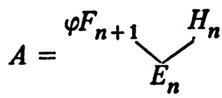


where $E_{n+1} = \tau\varphi(F_{n+1}) = \varphi_{n+1}(F_{n+1})$.

1.7. $f_\alpha(C_\alpha) \cap H_m = E_m$ for all $m \geqslant 0$. In particular, f_α is a proper embedding.

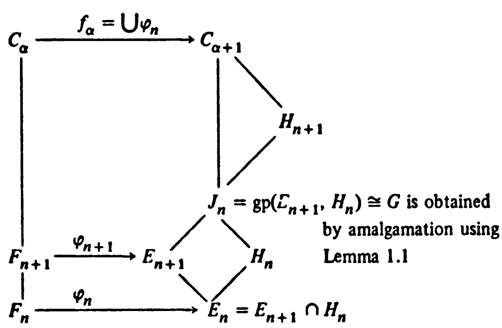
PROOF. Assume inductively $E_n \cap H_m = E_m$ for some $n > m$, $n = m$ being trivial. Then $E_{n+1} \cap H_m = E_{n+1} \cap H_n \cap H_m = E_n \cap H_m = E_m$ (using 1.6). Since $f_\alpha(C_\alpha) = \bigcup E_n$, 1.7 follows.

1.8. *Amalgamating property of* f_α. Let $J_n = \mathrm{gp}(E_{n+1}, H_n) < C_{\alpha+1}$ and let $G = G_n = \mathrm{gp}(A)$, where



and $\varphi \equiv f_\alpha$ on F_n, be obtained from Lemma 1.1. There is some $\tau \in \mathrm{ISO}(G_n, J_n)$ such that $\tau \equiv 1$ on H_n and $\tau\varphi \equiv f_\alpha$ on F_{n+1}. (The embedding τ defined above has these properties.)

These embeddings are represented in the following diagram in which slanted or vertical lines are inclusions:



1.9. *Summary of the construction of U.* A *ULF* group U of power $\aleph_1$ is obtained as the direct union of the groups $C_\alpha \cong C$ via the embeddings f_α: $C_\alpha \to C_{\alpha+1}$ for each $\alpha < \omega_1$ by imposing continuity at each limit ordinal.

Specifically, there exists a direct limit map $\sim$: $\bigcup\{C_\alpha\} \to U$ with the following properties. Let $U_\alpha = \tilde{C}_\alpha$.

1.10. $\sim$: $C_\alpha \to U_\alpha$ is an isomorphism and $U = \bigcup\{U_\alpha: \alpha < \omega_1\}$.

1.11. For all $x \in C_\alpha$, $\tilde{x} = \widetilde{f_\alpha(x)}$, and so $U_\alpha < U_{\alpha+1}$.

1.12 (continuity). If $\lambda > 0$ is a limit ordinal, then $U_\lambda = \bigcup\{U_\beta: \beta < \lambda\}$(1).

From 1.11 it follows that for all $\alpha < \omega_1$

1.13. $\tilde{F}_n = \tilde{E}_n$ (see 1.5), and

1.14. The embeddings $f_\alpha(C_\alpha) < C_{\alpha+1}$ and $U_\alpha < U_{\alpha+1}$ are isomorphic.

Each embedding f_α is determined by the choices made for (1) the chains $\{H_n^\alpha\} = \{H_n\}$ and $\{F_n^\alpha\} = \{F_n\}$ of finite subgroups of $C_{\alpha+1} = \bigcup\cdot H_n^\alpha$ and $C_\alpha = \bigcup F_n^\alpha$, (2) the amalgamating groups $G = G_n$ supplied by Lemma 1.1 as described in 1.8, and (3) the embeddings τ: $G_n \to C_{\alpha+1}$. The choices made for τ and for the chains $\{H_n^\alpha\}$ will be immaterial to the results of the paper.

We will see that the amalgamating group G can be chosen so that certain subgroups of G have a prescribed numerical structure, namely that any given prime dominates their order. It will follow that if $x \in U_{\alpha+1} - U_\alpha$, then $\mathrm{gp}(x, y)$ has this structure for almost all $y \in U_\alpha$; and this will give control over the numerical structure of all uncountable subgroups of U. This will be done in §2 where 0.1–0.3 will be proved. In §3 the effect of the chains $\{F_n^\alpha\}$ on the structure of U will be considered, and this will lead to the proof of 0.4.

2. Subgroup-incomparable *ULF* groups.

DEFINITION. The prime p *dominates* the finite group G if ${}^oP >$ index of P in G, where P is a p Sylow subgroup of G.

The next lemma prescribes the choice of the amalgamating groups G obtained from Lemma 1.1.

2.1. LEMMA. *Let p be a prime, $r > 1$, and $w_1, \ldots, w_d$ nontrivial words in a countable free group. There is a group G meeting the conditions in Lemma* 1.1 *such that, for all $x \in H - E$ and $y \in \varphi F - E$,* (i) *p dominates* $\mathrm{gp}(x, y)$, (ii) ${}^o\mathrm{gp}(x, y) > r$ *and* (iii) *if $x^2 \notin E$, then $w_i \not\equiv 1$ in* $\mathrm{gp}(x, y)$, $1 \leq i \leq d$.

PROOF. Let Q be a finite group with the properties of G in Lemma 1.1, so $Q = \mathrm{gp}(\varphi F, H)$, etc. Let $\mathcal{Q} = \varphi F *_E H$ (the amalgamated free product) and let $Q \cong \mathcal{Q}/K$ (naturally). K is a free group [7, p. 228, (5) and (6)] and is finitely generated since $\mathcal{Q}$ is f.g. and $\mathcal{Q}/K$ is finite [13, 8.4.13]. Using a well-known property of free groups, for every finite $S < K - 1$ there exists $P < K$ such that P is a characteristic subgroup of $\mathcal{Q}$, $S < K - P$, and K/P

(1) Note that $\sim$ is constructed inductively. $U_\lambda \cong C$ by 0.11 and the uniqueness of C; and $\sim$: $C_\lambda \to U_\lambda$ is an *arbitrary* isomorphism.

is a finite p group. The main results needed to prove this are [13, 7.1.6 and 8.4.16] and [1, Theorem 2.2].

Let $x \in H - E$ and $y \in \varphi F - E$. If $x^2 \notin E$, then xyx and yx generate a free subgroup of $\mathcal{G}$ [12, p. 510]. Hence,

2.2. If $x^2 \notin E$, then $K \cap \mathrm{gp}(x, y) = K_{xy}$ is a free group of rank > 1.
We can choose a characteristic subgroup $P = P(x, y)$ of $\mathcal{G}$ so that $P \leqslant K$, K/P is a finite p group, and, defining $\Phi = PK_{xy}/P \cong K_{xy}/P \cap \mathrm{gp}(x, y)$,

2.3. ${}^o\Phi > \max({}^oQ, r)$ and if $x^2 \notin E$, then $w_i \not\equiv 1$ in Φ, $1 \leqslant i \leqslant d$. (The first condition is met if $X \cup (XX^{-1} - 1) < K - P$ where $X < K_{xy} - 1$, ${}^oX > \max({}^oQ, r)$, while the second is met using 2.2 if $K_{xy} - P$ contains words in K_{xy} of structure similar to each w_i.)

Define $N = \bigcap\{P(x, y)\colon x \in H - E, y \in \varphi F - E\}$ and $G = \mathcal{G}/N$ and let $\beta\colon \mathcal{G} \to G$ be the factor map. G is finite because N is obtained as a finite intersection of subgroups of finite index. Let $\Omega = NK_{xy}/N < \beta\mathrm{gp}(x, y)$. Ω is a p group because K/N is. Since Φ is a factor of Ω, ${}^o\Omega > \max({}^oQ, r)$, and it follows that

2.4. p dominates $\beta\mathrm{gp}(x, y)$, because $\beta\mathrm{gp}(x, y)/\Omega \cong \mathrm{gp}(x, y)/K_{xy}$ and so has order $\leqslant {}^oQ$, and

2.5. ${}^o\beta\mathrm{gp}(x, y) > r$ and if $x^2 \notin E$, then $w_i \not\equiv 1$ in $\beta\mathrm{gp}(x, y)$, $1 \leqslant i \leqslant d$, (by 2.3 because Φ is a factor of Ω).

If we identify H with $\beta H < G$ and take $\beta\varphi\colon F \to G$ as the new φ, then the conclusions of our lemma follow from 2.4 and 2.5.

2.6. Definition of $\Pi(f_\alpha)$. Let Π be an infinite set of primes and let $f_\alpha\colon C_\alpha \to C_{\alpha+1}$ be the embedding of 1.9. The property $\Pi(f_\alpha)$ holds if (refer to 1.5) for all $n \geqslant 0$, $x \in H_n - E_n$, and $y \in E_{n+1} - E_n$, (1) p_n domaintes $\mathrm{gp}(x, y)$ where (p_i) is some list of Π, (2) ${}^o\mathrm{gp}(y, x) > |E_{n+1}|^2$, and (3) if $x^2 \notin E$, then $w_n \not\equiv 1$ in $\mathrm{gp}(y, x)$, $1 \leqslant i \leqslant n$, where (w_i) is some list of the nontrivial words in a countable free group.

2.7. Lemma. *The groups* G_n *of* 1.8 *can be chosen so that* $\Pi(f_\alpha)$ *holds.*

Proof. $G_n = \mathrm{gp}(A)$, where

$$A = \begin{array}{ccc} \varphi F_{n+1} & & H_n \\ & \searrow\ \swarrow & \\ & E_n & \end{array}$$

is obtained from Lemma 1.1. Lemma 2.1 allows us to choose G_n so that for all $x \in H_n - E_n$ and $y \in \varphi F_{n+1} - E_n$, p_n dominates $\mathrm{gp}(y, x)$, ${}^o\mathrm{gp}(y, x) > |E_{n+1}|^2 = r$, and if $x^2 \notin E_n$, then $w_i \not\equiv 1$ in $\mathrm{gp}(y, x)$, $1 \leqslant i \leqslant n$. Now the conclusions (1)–(3) follow by applying $\tau\colon G_n \to \mathrm{gp}(E_{n+1}, H_n)$.

2.8. Lemma. *Assume* $\Pi(f_\alpha)$ *holds. Let* $x \in C_{\alpha+1} - f_\alpha(C_\alpha)$ *and let* $Y \leqslant f_\alpha(C_\alpha)$ *be infinite. Then,*

(1) *$\{p$: p dominates* gp(y, x) *for some* $y \in Y\} = \Lambda \cup L$ *where* $\Lambda \leqslant \Pi$ *is infinite and L is finite,*

(2) gp(Y, x) *does not belong to a proper variety, and*

(3) *Define* $Y_i = \{y \in Y: p_i$ *dominates* gp$(y, x)\}$. *There is some* $n \geqslant 0$ *such that, for all* $i \geqslant n$, $Y_i = Y \cap (E_{i+1} - E_i)$ *and hence, for all* $j > n$, *we have*

2.9. ${}^oY \cap E_j = {}^oY \cap E_n + \sum_{i=n}^{j-1} {}^oY_i$.

PROOF. Refer to 1.5 and 1.8. Let $z \in C_{\alpha+1} - f_\alpha(C_\alpha)$. For some $m = m(z)$, $z \in H_m$. Let $y \in Y \cap (E_{j+1} - E_j)$ for some $j \geqslant m$. Since $z \in H_j - E_j$ we have by $\Pi(f_\alpha)$

2.10. p_j dominates gp(y, z), ^{o}gp$(y, z) > |E_{j+1}|^2$, and if $z^2 \notin E_j$, then $w_i \not\equiv 1$ in gp(y, z), $1 \leqslant i \leqslant j$. Recall that $p_j \in \Pi$.

To prove (1), define $\Lambda = \{p$: p dominates gp(y, x) for some $y \in Y - E_{m(x)}\}$. Taking $z = x$ in 2.10, we have $\Lambda \leqslant \Pi$. Since the groups E_i are finite, y can be chosen so that j is arbitrarily large. Hence Λ is infinite and (1) follows.

To prove (2), we first choose $y_0 \in Y - E_{m(x)}$ such that $y_0 \in E_{n+1} - E_n$ where $p_n \neq 2$. Taking $z = x$ and $j = n$ in 2.10, we have ^{o}gp$(y_0, x) > |E_{n+1}|^2$ and p_n dominates gp(y_0, x). It follows that no p_n Sylow subgroup of gp(y_0, x) is contained in E_{n+1}. So, there is a p_n-element $z \in$ gp$(x, y) - E_{n+1}$. Note that $z \in$ gp$(E_{n+1}, H_n) \leqslant H_{n+1}$ and thus $z, z^2 \notin \bigcup E_j = f_\alpha(C_\alpha)$ because of 1.7. Now we can apply 2.10: For arbitrarily large values of $j > m(z) = n + 1$, there is some $y \in Y \cap (E_{j+1} - E_j)$, and $w_i \not\equiv 1$ in gp(y, z) for all $i \leqslant j$. Hence gp$(Y, z) \leqslant$ gp(Y, x) belongs to no proper variety.

To prove (3), let $x \in H_m$ and define $P = \{p$: p dominates gp(y, x) for some $y \in Y \cap E_m\}$. Since P is finite we can choose $n > m$ so that, for all $i \geqslant n$, $p_i \notin P$. Suppose $i \geqslant n$ and $y \in Y_i$. We have $y \notin E_m$ because $p_i \notin P$, and so $y \in E_{r+1} - E_r$ for some $r \geqslant m$. Since $x \in H_r$, p_r dominates gp(y, x) by $\Pi(f_\alpha)$. Thus $r = i$ and we conclude $Y_i = Y \cap (E_{i+1} - E_i)$. The equality 2.9 is immediate because $Y \cap E_j$ is the disjoint union of the sets involved.

2.11. THEOREM. *There is a set* $\mathfrak{U}$ *of* $2^{\aleph_1}$ *ULF groups of order* $\aleph_1$ *satisfying* 0.1–0.3.

PROOF OF 0.1 AND 0.2. By a well-known set theoretic lemma [14, p. 451] there is a set $\mathfrak{P}$ such that each $\Pi \in \mathfrak{P}$ is an infinite set of primes, ${}^o\mathfrak{P} = \aleph_1$, and

2.12. If $\Pi \neq \Gamma \in \mathfrak{P}$, then $\Pi \cap \Gamma$ is finite.

Now apply the lemma again to obtain a set $\mathfrak{Q}$ such that, for each $\mathfrak{V} \in \mathfrak{Q}$, $\mathfrak{V} < \mathfrak{P}$, ${}^o\mathfrak{V} = \aleph_1$, ${}^o\mathfrak{Q} = 2^{\aleph_1}$, and

2.13. If $\mathfrak{V} \neq \mathfrak{T} \in \mathfrak{Q}$, then ${}^o\mathfrak{V} \cap \mathfrak{T} \leqslant \aleph_0$.

Let $\mathfrak{V}_0 \in \mathfrak{Q}$. Since ${}^o\mathfrak{V}_0 = \aleph_1$, let $\mathfrak{V}_0 = \{\Pi_\alpha: \alpha < \omega_1\}$. We now construct a *ULF* group $U = U_0$ as in 1.9 with the additional condition that $\Pi_\alpha(f_\alpha)$ holds for all $\alpha < \omega_1$. This is possible by Lemma 2.7.

Define $\mathfrak{U} = \{U_0: \mathcal{V}_0 \in \mathfrak{Q}\}$.

Let $U_0 = U \in \mathfrak{U}$. Recall (see 1.9) that $U = \bigcup\{U_\alpha: \alpha < \omega_1\}$, $U_\alpha \cong C$, so every countable subset of U is contained in some U_α. Let T be an uncountable subgroup of U and put $T_\alpha = T \cap U_\alpha$.

Let $Y < T$, ${}^oY = \aleph_0$. For each $x \in U$ define $P(Y, x) = \{p: p$ dominates $\mathrm{gp}(y, x)$ for some $y \in Y\}$. Consider the following property of a pair $(B, \mathcal{V})$, $B < U$, $\mathcal{V} \in \mathfrak{Q}$:

2.14. $Y \leqslant B < T$, ${}^oB = \aleph_0$, and for all $x \in T - B$ there is some $\Pi \in \mathcal{V}$ such that $P(Y, x) \cap \Pi$ is infinite.

We will prove two propositions:

2.15. *If* $B < U = U_0$ *and* $\mathcal{V} \in \mathfrak{Q}$ *satisfy* 2.14, *then* $\mathcal{V} = \mathcal{V}_0$.

2.16. *Let* β *be such that* $Y \leqslant T_\beta$. *Then* 2.14 *is satisfied if* $B = T_\beta$ *and* $\mathcal{V} = \mathcal{V}_0$.

PROOF OF 2.16. Let $x \in T - T_\beta$. Then, by continuity (1.12), $x \in T_{\alpha+1} - T_\alpha$ for some $\alpha \geqslant \beta$. Note $Y \leqslant T_\alpha$. Since the embeddings $U_\alpha < U_{\alpha+1}$ and $f_\alpha(C_\alpha) < C_{\alpha+1}$ are isomorphic (1.14) and $\Pi_\alpha(f_\alpha)$ holds, parts (1) and (2) of Lemma 2.8 imply

2.17. $P(Y, x) = \Lambda \cup L$ where $\Lambda < \Pi_\alpha$ is infinite and L is finite, and

2.18. $\mathrm{gp}(Y, x)$ does not belong to any proper variety.

Thus $P(Y, x) \cap \Pi_\alpha$ is infinite and 2.16 follows since $\Pi_\alpha \in \mathcal{V}_0$.

PROOF OF 2.15. Notice that, because of 2.12, 2.17 implies

2.19. If $x \in T - T_\beta$, then Π_α is the unique $\Pi \in \mathcal{P}$ such that $P(Y, x) \cap \Pi$ is infinite where $x \in T_{\alpha+1} - T_\alpha$.

Now suppose $B < U_0$ and $\mathcal{V} \in \mathfrak{Q}$ satisfy 2.14. Let $Y \leqslant T_\beta$ and let $x \in T - (B \cup T_\beta)$. By 2.19 we have

2.20. $\Pi_\alpha \in \mathcal{V}_0 \cap \mathcal{V}$ where $x \in T_{\alpha+1} - T_\alpha$.

Since ${}^oT = \aleph_1$, 2.20 must hold for an unbounded set of $\alpha < \omega_1$, which implies ${}^o\mathcal{V}_0 \cap \mathcal{V} = \aleph_1$, and from 2.13 we now conclude $\mathcal{V}_0 = \mathcal{V}$ as desired.

It follows that $\mathcal{V}_0$ is determined uniquely by the condition 2.14 from the structure of any uncountable subgroup T of U_0, and this proves 0.1. Note that 0.2 is immediate from 2.18 because $\mathrm{gp}(Y, x) < T$.

PROOF OF 0.3. Let $U \in \mathfrak{U}$ and let $S \leqslant T$ be uncountable subgroups of U. Suppose there exists $\sigma \in \mathrm{ISO}(S, T)$. Let $S_\alpha = U_\alpha \cap S$. By the continuity of the chain $\{U_\alpha\}$, there is some β such that $\sigma S_\beta = T_\beta$ and T_β is infinite. Let $\alpha \geqslant \beta$. By 2.19, letting $Y = T_\beta$, we have $T_{\alpha+1} - T_\alpha = \{x \in T - T_\beta: P(T_\beta, x) \cap \Pi_\alpha$ is infinite$\}$, and, replacing T by S, we likewise have $S_{\alpha+1} - S_\alpha = \{x \in S - S_\beta: P(S_\beta, x) \cap \Pi_\alpha$ is infinite$\}$. Since $\sigma S_\beta = T_\beta$ it follows that $\sigma(S_{\alpha+1} - S_\alpha) = T_{\alpha+1} - T_\alpha$, and

2.21. $\sigma S_\alpha = T_\alpha$ for all $\alpha \geqslant \beta$.

Suppose $\alpha \geqslant \beta$ and there exists $z \in S_{\alpha+1} - S_\alpha$. Let $x = \sigma(z) \in T_{\alpha+1} - T_\alpha$. Recall that $U_\alpha = \bigcup \tilde{E}_n$ (see 1.9). Let $Y = T_\alpha$ in Lemma 2.8; part (3) implies that $n = n_1$ exists such that, for all $j > n$, ${}^oT_\alpha \cap \tilde{E}_j = {}^oT_\alpha \cap \tilde{E}_n + \Sigma_{i=n}^{j-1} {}^oY_i$ where $Y_i = \{y \in T_\alpha: p_i \text{ dominates } \mathrm{gp}(y, x)\}$. Putting $Y = S_\alpha$ gives the existence of $n = n_2$ such that, for all $j > n$, ${}^oS_\alpha \cap \tilde{E}_j = {}^oS_\alpha \cap \tilde{E}_n + \Sigma_{i=n}^{j-1} {}^oY_i'$ where $Y_i' = \{y \in S_\alpha: p_i \text{ dominates } \mathrm{gp}(y, z)\}$. Since $\sigma S_\alpha = T_\alpha$ and $\sigma(z) = x$, we have $\sigma Y_i = Y_i'$, and so ${}^oY_i = {}^oY_i'$, for all i. Hence, if $n = \max(n_1, n_2)$ we conclude

2.22. $|{}^oT_\alpha \cap \tilde{E}_j - {}^oS_\alpha \cap \tilde{E}_j| \leqslant {}^oE_n$ for all $j > n$.

(The fact that n can be increased in 2.9 follows from the full statement of (3).) Assume $S_\alpha < T_\alpha$. Since $S_\alpha = \bigcup \{S_\alpha \cap \tilde{E}_j\}$ is infinite, there is some $j > n$ such that ${}^oS_\alpha \cap \tilde{E}_j > {}^oE_n$ and $S_\alpha \cap \tilde{E}_j < T_\alpha \cap \tilde{E}_j$, and this contradicts 2.22. Hence $S_\alpha = T_\alpha$ for an unbounded set of $\alpha < \omega_1$, and so $S = T$.

3. **Complete** *ULF* **groups.** The next lemma shows that the numerical structure of the groups U constructed in Theorem 2.11 places a severe restriction on their automorphisms.

3.1. LEMMA. *Let* $\mathcal{U}$ *be the set of groups constructed in Theorem* 2.11. *Let* $U \in \mathcal{U}$ *and* $\theta \in \mathrm{Aut}\, U$. (a) *Let* $\tau < \omega_1$ *be minimal such that* $\theta U_\tau = U_\tau$. *Then, for all* $\alpha \geqslant \tau$, $\theta U_\alpha = U_\alpha$; (b) *Suppose* $\alpha \geqslant \lambda \geqslant \tau$. *Using the notation of* 1.5 *and* 1.9, *we have* $U_\alpha = \bigcup \tilde{E}_n$ *and* $U_{\alpha+1} = \bigcup \tilde{H}_n$. *Let* $D_n^\alpha = D_n = U_\lambda \cap \tilde{E}_n$. *Then* $\theta D_n = D_n$ *for all but finitely many values of n. In particular, taking* $\lambda = \alpha$, $\theta \tilde{E}_n = \tilde{E}_n$ *for almost all n.*

PROOF OF (a). This is actually a special case of 2.21, letting $S = T = U$, $\beta = \tau$, and $\sigma = \theta$.

PROOF OF (b). Choose any $x \in U_{\alpha+1} - U_\alpha$. Then $\theta(x) \in U_{\alpha+1} - U_\alpha$ by part (a). Choose m so that $x, \theta(x) \in \tilde{H}_m$. Suppose $\theta D_n \neq D_n$ for infinitely many n. Then the set $\{a \in U_\alpha: a \in D_{j+1} - D_j \text{ and } \theta(a) \notin D_{j+1}\}$ is infinite, and we can choose $j \geqslant m$ and $a \in D_{j+1} - D_j$ so that $\theta(a) \notin D_{j+1}$. By part (a), $\theta(a) \in D_{n+1} - D_n$ for some $n > j$. Since $\Pi_\alpha(f_\alpha)$ holds we have (1) p_j dominates $\mathrm{gp}(a, x)$ because $a \in \tilde{E}_{j+1} - \tilde{E}_j$ and $x \in \tilde{H}_j - \tilde{E}_j$, and similarly (2) p_n dominates $\mathrm{gp}(\theta(a), \theta(x))$ where p_j and p_n are the jth and nth members of a list of Π_α. Since $\theta \in \mathrm{Aut}\, U_{\alpha+1}$ we must have $p_j = p_n$, and this contradiction proves (b).

The next lemma gives a convenient criterion for the completeness of a group U.

3.2. LEMMA. *Refer to* 1.10. *Suppose* $\theta \in \mathrm{Aut}\, U$. *Let* θ_α = *the restriction of* θ *to* U_α. *Suppose that for all limit ordinals* $\lambda < \omega_1$ *such that* $\theta U_\lambda = U_\lambda$ *we have* $\theta_\lambda \in \mathrm{Inn}\, U_\lambda$. *Then* $\theta \in \mathrm{Inn}\, U$.

The proof uses the well-known regressive function principle for ω_1. Continuity of the chain $\{U_\alpha\}$ (1.12) guarantees that $\theta U_\lambda = U_\lambda$ for all λ belonging to a closed, unbounded subset of ω_1. For each such λ, choose

$x_\lambda \in U_\lambda$ such that $\theta_\lambda =$ conjugation by x_λ. According to the regressive function principle, the function $\lambda \to x_\lambda$ is constant on some unbounded subset S of ω_1, that is, $x_\lambda = x$ for all $\lambda \in S$. So $\theta =$ conjugation by x. More details can be found in [4, Lemma 3.3 and the proof of Theorem 6.5] or, more appropriately, [2].

Lemmas 3.1 and 3.2 provide a method to obtain complete groups $U \in \mathfrak{U}$: For each limit group U_λ, $\lambda \geq \tau$, we would like the conditions in 3.1(b), namely that $\theta D_n^\alpha = D_n^\alpha$ for all $\alpha \geq \lambda$ and for almost all n to guarantee that $\theta_\lambda \in \operatorname{Inn} U_\lambda$. Since $D_n^\alpha = U_\lambda \cap \tilde{F}_n^\alpha$ (see 1.5 and 1.13), the chains $\{D_n^\alpha\}$ depend on the chains $\{F_n^\alpha\}$, $\alpha \geq \lambda$, which we are free to choose as we please (see 1.9). It is not yet clear exactly how many chains $\{D_n^\alpha\}$ (for various $\alpha \geq \lambda$) must be utilized to smother the outer automorphisms of U_λ. There might in fact be a single chain $\{D_n^\lambda\} = \{\tilde{F}_n^\lambda\}$ which suffices by itself: that is, if $\theta \tilde{F}_n^\lambda = \tilde{F}_n^\lambda$ for almost all n, then $\theta_\lambda \in \operatorname{Inn} U_\lambda$. However, the present proof will use two chains $\{\tilde{F}_n^\lambda\}$ and $\{D_n^{\lambda+1}\} = \{U_\lambda \cap \tilde{F}_n^{\lambda+1}\} = \{\tilde{I}_n^\lambda\}$. These chains will now be defined.

3.3. Definition of the chains $\{F_i^\lambda\}$ and $\{I_i^\lambda\}$ of C_λ. Recalling Hall's construction of C (see Introduction), let $C_\lambda = \bigcup L_n$ where L_1 is a symmetric group of degree 3, $L_{n+1} \cong$ the symmetric group on L_n and $L_n < L_{n+1}$ is the left regular embedding. Let $R_n =$ the centralizer of L_n in L_{n+1}. It is easily shown that R_n is the right regular representation of L_n in L_{n+1}. So $\operatorname{gp}(L_n, R_n) = L_n \oplus R_n < L_{n+1}$. Since R_n is a symmetric group of degree $d \geq 3$, there is a chain $R_n^1 < \cdots < R_n^{d-2} = R_n$ where R_n^i is the subgroup fixing each of $d-(i+2)$ permuted objects. Note $R_n^1 \cong L_1$ and let $A_n \neq B_n < R_n^1$ be subgroups of order 2. The chain $\{F_i^\lambda\}$ will consist of all the members of segments of the form $\cdots L_n < L_n \oplus A_n < L_n \oplus R_n^1 < \cdots < L_n \oplus R_n^{d-2} < L_{n+1} \cdots$ $(n \geq 1)$. The chain $\{I_i^\lambda\}$ is defined identically, except we *replace each A_n by B_n* in the second member of each segment. Note that $\{L_n\}$ is a subchain of both $\{F_n^\lambda\}$ and $\{I_n^\lambda\}$.

3.4. Lemma. *If $\Gamma \in \operatorname{Aut} C_\lambda$ and almost all the members of the chains $\{F_i^\lambda\} = \{F_i\}$ and $\{I_i^\lambda\} = \{I_i\}$ are invariant under Γ, then $\Gamma \in \operatorname{Inn} C_\lambda$.*

Proof. The proof depends on the completeness of the finite symmetric group [13, 11.4]. Let m be such that for all $i \geq m$, F_i and I_i are invariant under Γ. Choose n so that $F_m \leq L_n$. Thus $\Gamma L_n = L_n$ and since L_n is a complete group, there is some $x \in L_n$ such that

3.5. $\Gamma(a) = x^{-1}ax$ for all $a \in L_n$.

We will show by induction that 3.5 is valid for all $a \in C_\lambda$. Assume that 3.5 holds for all $a \in L_j, j \geq n$. Since $\Gamma L_{j+1} = L_{j+1}$ and L_{j+1} is complete, for some $y \in L_{j+1}$, $\Gamma(a) = y^{-1}ay$ for all $a \in L_{j+1}$. Hence $z = yx^{-1}$ centralizes L_j, implying $z \in R_j$. All of the groups in the series $L_j < L_j \oplus A_j$ or $L_j \oplus B_j <$

$L_j \oplus R_j^1 < \cdots < L_j \oplus R_j^{d-2}$ are invariant under Γ and hence under conjugation by z, and it follows (e.g. by Krull-Schmidt) that each of the groups $A_j, B_j, R_j^1, \ldots, R_j^{d-2} = R_j$ are also z-invariant. However, from the definitions of these subgroups (see 3.3), it is seen that only the identity of R_j normalizes all of them. Consequently, $z = 1$, $x = y$, and the induction is complete.

We can now give the condition for the completeness of U.

3.6. Lemma. *Let* $U \in \mathfrak{U}$ = *the set of groups constructed in Theorem* 2.11. *Suppose that for all limit* $\lambda < \omega_1$ *the chain* $\{F_i^\lambda\}$ *is defined as in* 3.3 *and the chain* $\{F_i^{\lambda+1}\}$ *is chosen so that*

3.7. $\tilde{I}_i^\lambda = U_\lambda \cap \tilde{F}_i^{\lambda+1} (= D_i^{\lambda+1})$ *for all* $i \geqslant 0$.

Then Aut U = Inn U.

Proof. Let $\theta \in$ Aut U. Suppose λ is a limit ordinal and $\theta U_\lambda = U_\lambda$. By Lemma 3.1(b), letting $\alpha = \lambda$ and $\alpha = \lambda + 1$ respectively, we have $\theta \tilde{F}_i^\lambda = \tilde{F}_i^\lambda$ and $\theta D_i^{\lambda+1} = D_i^{\lambda+1}$, and so by 3.7 $\theta \tilde{I}_i^\lambda = \tilde{I}_i^\lambda$ for almost all $i \geqslant 0$. Now Lemma 3.4 (via $\sim$) implies $\theta_\lambda \in$ Inn U_λ and, by Lemma 3.2, $\theta \in$ Inn U.

All that remains is to define the chain $\{F_i^{\lambda+1}\}$ in such a way that 3.7 holds.

We will use the notations of 1.5, 1.8, and 1.9 as follows. $C_{\lambda+1} = \bigcup H_i = \bigcup F_i^{\lambda+1}$, $C_\lambda = \bigcup F_i^\lambda$, $f_\lambda(F_i^\lambda) = E_i < C_{\lambda+1}$, and $\mathrm{gp}(E_{i+1}, H_i) = J_i$. Note that

3.8. J_i is generated by the amalgam

$$A_i = \begin{array}{ccc} E_{i+1} & & H_i \\ & \searrow\!\!\swarrow & \\ & E_i & \end{array}$$

and is obtained (see 1.8) isomorphically from Lemma 2.1 (because of the construction of $\mathfrak{U}$ in 2.11), and

3.9. $f_\lambda(C_\lambda) \cap H_i = E_i$ (by 1.7).

The desired condition 3.7 is equivalent to

3.10. $f_\lambda(I_i^\lambda) = f_\lambda(C_\lambda) \cap F_i^{\lambda+1}$ (see 1.11).

There are two cases to be considered in defining $F_i^{\lambda+1}$.

Case I. F_i^λ has the form L_n or $L_n \oplus R_n^j$, $1 \leqslant j \leqslant d-2$. In this case we define $F_i^{\lambda+1} = H_i$ and 3.10 is satisfied, using 3.9, because $I_i^\lambda = F_i^\lambda$ by Definition 3.3.

Case II. F_i^λ has the form $L_n \oplus A_n$. In this case $F_{i-1}^\lambda = L_n$ and $I_i^\lambda = L_n \oplus B_n < L_n \oplus R_n^1 = F_{i+1}^\lambda$. $F_i^{\lambda+1}$ must contain $H_{i-1} = F_{i-1}^{\lambda+1}$ (by Case I) and so, to satisfy 3.10, it is natural to define

3.11. $F_i^{\lambda+1}$ is generated by the "subamalgam"

$$A' = \begin{array}{ccc} f_\lambda(L_n \oplus B_n) & & H_{i-1} \\ & \searrow\!\!\swarrow & \\ & E_{i-1} & \end{array}$$

of the amalgam A_i of 3.8.

(For the definition of subamalgam see 3.13. That A' is a subamalgam of A_i follows from 3.9 and the fact $f_\lambda(L_n \oplus B_n) \cap E_i = f_\lambda(L_n \oplus B_n) \cap f_\lambda(L_n \oplus A_n) = f_\lambda(L_n) = E_{i-1}$.) Since $A_i < H_{i+1}$, we have

$$F_i^{\lambda+1} \cap f_\lambda(C_\lambda) < H_{i+1} \cap f_\lambda(C_\lambda) = E_{i+1} \quad \text{by 3.9,}$$

and so

$$F_i^{\lambda+1} \cap f_\lambda(C_\lambda) = F_i^{\lambda+1} \cap E_{i+1}.$$

Hence 3.10 will be satisfied just in case

3.12. $f_\lambda(L_n \oplus B_n) = F_i^{\lambda+1} \cap E_{i+1}$.

This condition will hold provided the group $J_i = \text{gp}(A_i)$ of 3.8 has a certain "subamalgam property".

3.13. Definition of the subamalgam property $S(A, G)$. If

$$A = \begin{array}{ccc} F & & H \\ & \diagdown\diagup & \\ & E & \end{array}$$

is an amalgam, then $A_0 \leqslant A$ is a *subamalgam of A* if

$$A_0 = \begin{array}{ccc} F_0 & & H_0 \\ & \diagdown\diagup & \\ & E_0 & \end{array}$$

where $F_0 \cap E = H_0 \cap E = E_0$, $F_0 \leqslant F$, and $H_0 \leqslant H$.

The *subamalgam property* $S(A, G)$ holds if $G = \text{gp}(A)$ and, for every subamalgam A_0 of A, we have $\text{gp}(A_0) \cap A = A_0$.

So far we have observed

3.14. Lemma *If* $S(A_i, J_i)$ *holds for all i, where* A_i *is the amalgam of* 3.8, *then* 3.12 $\Rightarrow$ 3.10 $\Rightarrow$ 3.7 *all hold, and* $U \in \mathfrak{U}$ *is a complete group by Lemma* 3.6.

Proof. 3.12 follows directly from 3.11 and the property $S(A_i, J_i)$.

To insure that $S(A_i, J_i)$ holds it will be necessary, in view of 3.8, to obtain groups G from Lemma 2.1 such that $S(A, G)$ holds where

$$A = \begin{array}{ccc} \varphi F & & H \\ & \diagdown\diagup & \\ & E & \end{array}$$

The following definition will facilitate the proof of this: Suppose $G = \text{gp}_1(A)$ and $Q = \text{gp}_2(A)$ where

$$A = \begin{array}{ccc} F & & H \\ & \diagdown\diagup & \\ & E & \end{array}.$$

The notation $\Psi\colon G \xrightarrow{A} Q$ will mean that Ψ is a homomorphism and $\Psi(a) = a$ for all $a \in A$.

3.15. Lemma. *Using the above notation,*

(a) *Suppose* A_0 *is a subamalgam of* A, $\mathrm{gp}_2(A_0) \cap A = A_0$, *and* $\Psi\colon G \xrightarrow{A} Q$. *Then* $\mathrm{gp}_1(A_0) \cap A = A_0$.

(b) *Suppose* $S(A, Q)$ *holds and* $\Psi\colon G \xrightarrow{A} Q$. *Then* $S(A, G)$ *holds.*

The proof is straightforward.

3.16. LEMMA. (a) *Recall that in the proof of Lemma* 2.1 *we began with any group* $Q = \mathrm{gp}_1(A)$,

$$A = \begin{smallmatrix} \varphi F & & H \\ & \diagdown\ \diagup & \\ & E & \end{smallmatrix},$$

and we then constructed a group $G = \mathrm{gp}_2(A)$ *satisfying* (i)–(iii). *There is a map* $\Psi\colon G \xrightarrow{A} Q$.

(b) *Suppose* A *is a finite amalgam and for each subamalgam* A_0 *of* A *there is a finite group* $G_0 = \mathrm{gp}_0(A)$ *such that* $\mathrm{gp}_0(A_0) \cap A = A_0$. *Then there is a finite group* $G = \mathrm{gp}(A)$ *such that* $S(A, G)$ *holds.*

PROOF OF (a). Recall that $Q = (\varphi F *_E H)/K$ and $G = (\varphi F *_E H)/N$, $N < K$, where both H and φF are identified with their images mod K and N. If Ψ is the factor map, then $\Psi\colon\ G \xrightarrow{A} Q$.

PROOF OF (b) Let

$$A = \begin{smallmatrix} F & & H \\ & \diagdown\ \diagup & \\ & E & \end{smallmatrix}.$$

For each A_0 let $G_0 = (F *_E H)/N_0$ and define $N = \bigcap\{N_0\colon A_0 < A\}$ and $G = (F *_E H)/N$. G is finite since there are only finitely many A_0. For each A_0, Ψ_0: $G \xrightarrow{A} G_0$ where Ψ_0 is the factor map, and by 3.15 (a) we conclude $S(A, G)$ holds.

3.17. LEMMA. *If for every finite amalgam*

$$A = \begin{smallmatrix} \varphi F & & H \\ & \diagdown\ \diagup & \\ & E & \end{smallmatrix}$$

there is a finite $Q = \mathrm{gp}(A)$ *such that* $S(A, Q)$ *holds, then the groups* G *of Lemma* 2.1 *can be obtained so that* $S(A_i, J_i)$ *holds for all* i. *With these choices for* G *Lemma* 3.14 *implies that* $U \in \mathfrak{U}$ *is a complete group.*

PROOF. To obtain the group G of Lemma 2.1, we can begin with a group Q such that $S(A, Q)$ holds. Then, by 3.16 (a) and 3.15 (b), $S(A, G)$ also holds. Since $J_i = \mathrm{gp}(A_i)$ is obtained isomorphically from $G = \mathrm{gp}(A)$ (see 3.8 and 1.8), we have $S(A_i, J_i)$ also.

All that is left is to fulfil the hypothesis of Lemma 3.17.

THEOREM. *If*

$$A = \begin{smallmatrix} F & & H \\ & \searrow\swarrow & \\ & E & \end{smallmatrix}$$

is a finite amalgam, there is a finite group $G = \mathrm{gp}(A)$ *such that* $S(A, G)$ *holds.*

PROOF. By Lemma 3.16 (b), we need to show that if

$$A_0 = \begin{smallmatrix} F_0 & & H_0 \\ & \searrow\swarrow & \\ & E_0 & \end{smallmatrix}$$

there is a finite group $G = \mathrm{gp}(A)$ such that $\mathrm{gp}(A_0) \cap A = A_0$.

We will use the notation of [11, §3] with A, B and H replaced by F, H and E respectively, that is, G will be constructed as a permutational product of F and H with a specific choice of left transversals, S for E in F, and T for E in H. We will choose S and T so that $S_0 \leqslant S$ where S_0 is a transversal for E_0 in F_0 and $T_0 \leqslant T$ where T_0 is a transversal for E_0 in H_0. Now $G = \mathrm{gp}(\rho F, \rho H)$ where each $\rho(a)$, $a \in F$, and $\rho(b)$, $b \in H$, is a permutation of the set $S \times T \times E$ induced by translation: $(s, t, e)\rho(a) = (s', t, e')$ where $sea = s'e'$ and $(s, t, e)\rho(b) = (s, t', e')$ where $teb = t'e'$. It follows that every permutation in $\mathrm{gp}(\rho F_0, \rho H_0)$ permutes the triples $S_0 \times T_0 \times E_0$ among themselves, but that no element of $\rho(F - F_0)$ or $\rho(H - H_0)$ has this property. Identifying F with ρF and H with ρH, the theorem is proved.

SOME QUESTIONS. Apart from the obvious question of extending these results to $\kappa > \aleph_1$, there is a question whether the present technique can be used to construct *ULF* groups of order $\aleph_1$ satisfying more stringent conditions; and, somewhat closer to home, there are interesting problems concerning the structure of Hall's group C.

It would be interesting to see if the following questions can be approached by refining the choice of the amalgamating groups and the chains in this construction. *Does there exist a ULF group of order* $\aleph_1$ *all of whose uncountable subgroups are* (*simple, complete, universal, etc.*).

It was mentioned in the paragraph before 3.3 that some single chain $\{F_i^\lambda\}$ might suffice to smother the outer automorphisms of C. A possibility for such a chain is $\{D_i\}$ where D_{i+1} is the symmetric group on D_i and D_i is embedded in D_{i+1} as the diagonal of $R \oplus L$, the direct sum of the right and left regular representations of D_i in D_{i+1}. It can be shown that the centralizer of D_i in D_{i+1} is 1, and so the proof of Lemma 3.4 would show that any automorphism of $D = \bigcup D_i$ which stabilizes the chain $\{D_i\}$ is inner. *Is* $D \cong C$? If this could be shown, the last part of our proof would be unnecessary.

The automorphism group of C might also merit some attention. We mentioned already that Aut C contains a free group on $2^{\aleph_0}$ generators. *Does*

Aut C *contain the symmetric group on a countable set? Is there an infinite set* $S < C$ *such that every permutation of* S *can be lifted to an automorphism of* C? Finally, we mention an interesting question of J. E. Roseblade. *Does* Inn C = *the locally finite radical of* Aut C?

ADDED IN PROOF. Since this paper was written, S. Shelah has kindly shown us an elegant and compact "tree-limit" construction whereby $2^{\aleph_0}$ subgroup-incomparable *ULF* groups of power $2^{\aleph_0}$ are obtained. The property 0.2 is also met, and he conjectures that complete groups can also be obtained. His method also applies to e.c. groups and skew fields. The property 0.3 does not hold for the groups of Shelah's construction, but we could hardly expect it to, since, e.g., every e.c. group of power κ possesses a subgroup of power κ which contains a proper copy of itself, namely the centralizer of an element of infinite order.

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